



HAL
open science

Algebraicity and transcendence of power series: combinatorial and computational aspects

Alin Bostan

► **To cite this version:**

Alin Bostan. Algebraicity and transcendence of power series: combinatorial and computational aspects. Doctoral. 3rd Algorithmic and Enumerative Combinatorics Summer School, Hagenberg, 2016 Austria. 2016. cel-01391907

HAL Id: cel-01391907

<https://inria.hal.science/cel-01391907v1>

Submitted on 4 Nov 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Algebraicity and transcendence of power series: combinatorial and computational aspects

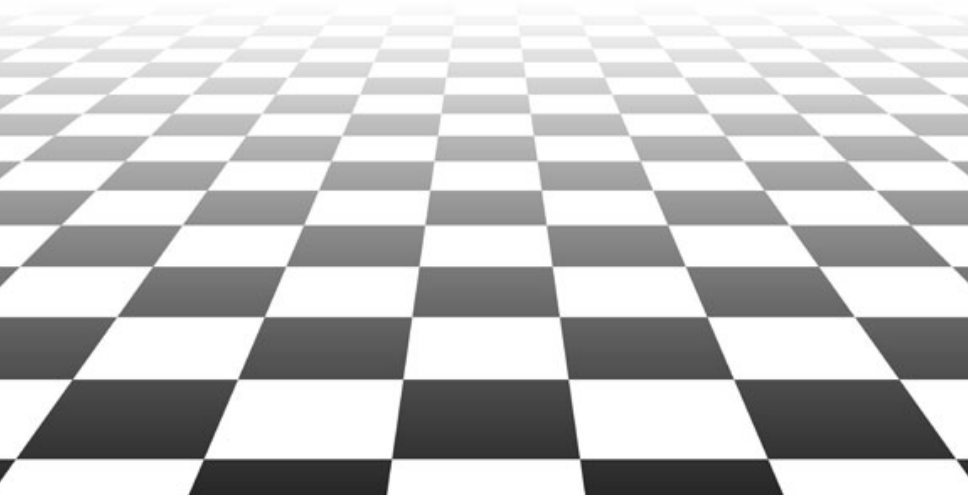
Alin Bostan



Algorithmic and Enumerative Combinatorics
RISC, Hagenberg, August 1–5, 2016

- ① Monday: Context and Examples
- ② Tuesday: Properties and Criteria (1)
- ③ Wednesday: Properties and Criteria (2)
- ④ Thursday: Algorithmic Proofs of Algebraicity
- ⑤ Friday: Transcendence in Lattice Path Combinatorics

Part V: Transcendence in Lattice Path Combinatorics

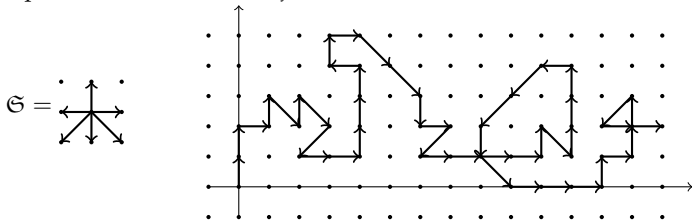


Lattice walks with small steps in the quarter plane

- ▷ We focus on **nearest-neighbor walks in the quarter plane**, i.e. walks in \mathbb{N}^2 starting at $(0,0)$ and using steps in a *fixed* subset \mathfrak{S} of

$$\{\swarrow, \leftarrow, \nearrow, \uparrow, \rightarrow, \searrow, \downarrow\}.$$

- ▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

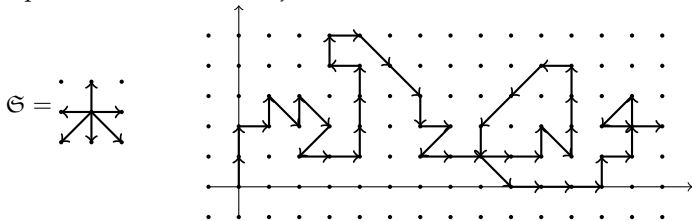


Lattice walks with small steps in the quarter plane

- ▷ We focus on **nearest-neighbor walks in the quarter plane**, i.e. walks in \mathbb{N}^2 starting at $(0,0)$ and using steps in a *fixed* subset \mathfrak{S} of

$$\{\swarrow, \leftarrow, \nearrow, \uparrow, \rightarrow, \searrow, \downarrow\}.$$

- ▷ Example with $n = 45$, $i = 14$, $j = 2$ for:



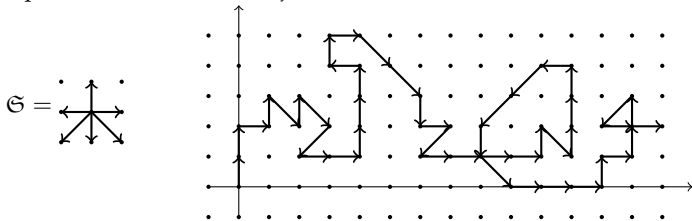
- ▷ Counting sequence: $f_{n;i,j}$ = number of **walks of length n ending at (i,j)** .

Lattice walks with small steps in the quarter plane

- ▷ We focus on **nearest-neighbor walks in the quarter plane**, i.e. walks in \mathbb{N}^2 starting at $(0,0)$ and using steps in a *fixed* subset \mathfrak{S} of

$$\{\swarrow, \leftarrow, \nearrow, \uparrow, \rightarrow, \searrow, \downarrow\}.$$

- ▷ Example with $n = 45$, $i = 14$, $j = 2$ for:



- ▷ Counting sequence: $f_{n;i,j}$ = number of **walks of length n ending at (i,j)** .
- ▷ Specializations:
- $f_{n;0,0}$ = number of **walks of length n returning to origin** (“excursions”);
 - $f_n = \sum_{i,j \geq 0} f_{n;i,j}$ = number of **walks with prescribed length n** .

▷ Complete generating series:

$$F(t; x, y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]].$$

▷ Complete generating series:

$$F(t; x, y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]].$$

▷ Specializations:

- Walks returning to the origin (“excursions”):

$$F(t; 0, 0);$$

- Walks with prescribed length:

$$F(t; 1, 1) = \sum_{n \geq 0} f_n t^n;$$

- Walks ending on the horizontal axis:

$$F(t; 1, 0);$$

- Walks ending on the diagonal:

$$“F(t; 0, \infty)” := [x^0] F(t; x, 1/x).$$

▷ Complete generating series:

$$F(t; x, y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]].$$

▷ Specializations:

- Walks returning to the origin (“excursions”): $F(t; 0, 0)$;
- Walks with prescribed length: $F(t; 1, 1) = \sum_{n \geq 0} f_n t^n$;
- Walks ending on the horizontal axis: $F(t; 1, 0)$;
- Walks ending on the diagonal: $“F(t; 0, \infty)” := [x^0] F(t; x, 1/x)$.

Combinatorial questions:

Given \mathfrak{S} , what can be said about $F(t; x, y)$, resp. $f_{n,i,j}$, and their variants?

- **Structure** of F : algebraic? transcendental?
- **Explicit form**: of F ? of f ?
- **Asymptotics** of f ?

Generating series and combinatorial problems

▷ Complete generating series:

$$F(t; x, y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]].$$

▷ Specializations:

- Walks returning to the origin (“excursions”): $F(t; 0, 0)$;
- Walks with prescribed length: $F(t; 1, 1) = \sum_{n \geq 0} f_n t^n$;
- Walks ending on the horizontal axis: $F(t; 1, 0)$;
- Walks ending on the diagonal: $“F(t; 0, \infty)” := [x^0] F(t; x, 1/x)$.

Combinatorial questions:

Given \mathfrak{S} , what can be said about $F(t; x, y)$, resp. $f_{n,i,j}$, and their variants?

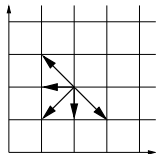
- **Structure** of F : algebraic? transcendental?
- **Explicit form**: of F ? of f ?
- **Asymptotics** of f ?

Our goal: Use computer algebra to give computational answers.

From the 2^8 step sets $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:

Small-step models of interest

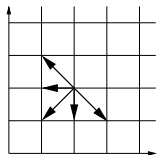
From the 2^8 step sets $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:



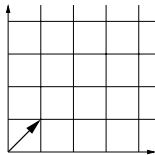
trivial,

Small-step models of interest

From the 2^8 step sets $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:



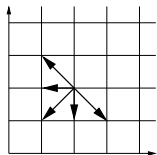
trivial,



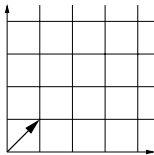
simple,

Small-step models of interest

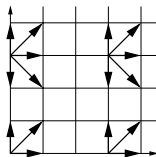
From the 2^8 step sets $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:



trivial,



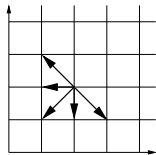
simple,



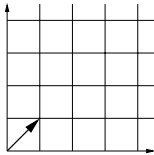
intrinsic to the
half plane,

Small-step models of interest

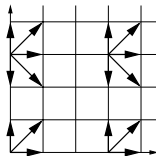
From the 2^8 step sets $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:



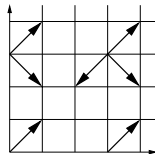
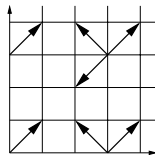
trivial,



simple,



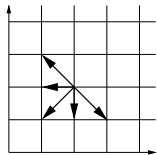
intrinsic to the
half plane,



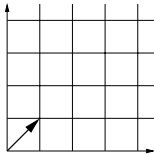
symmetrical.

Small-step models of interest

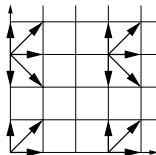
From the 2^8 step sets $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:



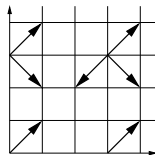
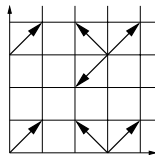
trivial,



simple,



intrinsic to the
half plane,

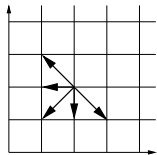


symmetrical.

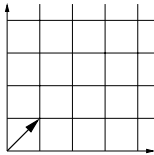
One is left with [79 interesting distinct models](#).

Small-step models of interest

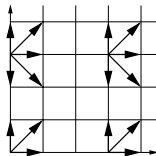
From the 2^8 step sets $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:



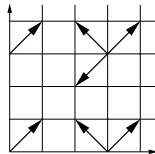
trivial,



simple,



intrinsic to the
half plane,



symmetrical.

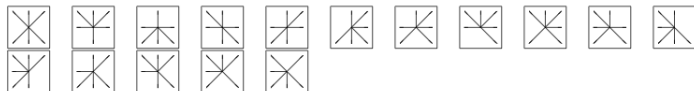
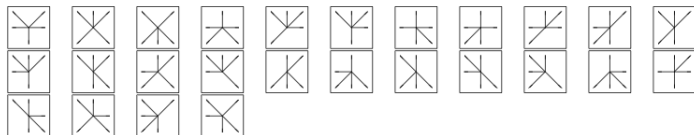
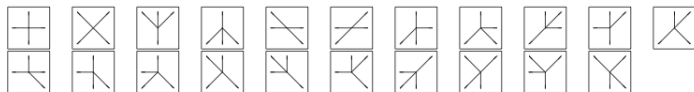
One is left with [79 interesting distinct models](#).

Is any further classification possible?

The 79 models



Non-singular



Singular

The 79 models





Non-singular




Singular

“Special” models

Dyck: 

Motzkin: 

Pólya: 

Kreweras: 

Gessel: 

Gouyou-Beauchamps: 

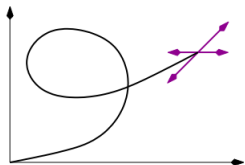
King: 

Algebraic reformulation: solving a functional equation

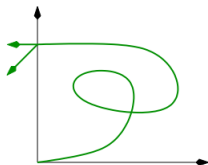
Generating function: $G(t; x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^n \sum_{j=0}^n g(n; i, j) t^n x^i y^j \in \mathbb{Q}[x, y][[t]]$

“Kernel equation”:

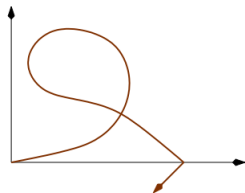
$$G(t; x, y) = 1 + t \left(xy + x + \frac{1}{xy} + \frac{1}{x} \right) G(t; x, y) \\ - t \left(\frac{1}{x} + \frac{1}{x} \frac{1}{y} \right) G(t; 0, y) - t \frac{1}{xy} (G(t; x, 0) - G(t; 0, 0))$$



⊖



⊖

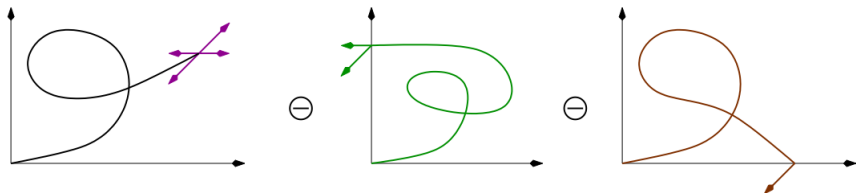


Algebraic reformulation: solving a functional equation

Generating function: $G(t; x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^n \sum_{j=0}^n g(n; i, j) t^n x^i y^j \in \mathbb{Q}[x, y][[t]]$

“Kernel equation”:

$$G(t; x, y) = 1 + t \left(xy + x + \frac{1}{xy} + \frac{1}{x} \right) G(t; x, y) \\ - t \left(\frac{1}{x} + \frac{1}{x} \frac{1}{y} \right) G(t; 0, y) - t \frac{1}{xy} (G(t; x, 0) - G(t; 0, 0))$$



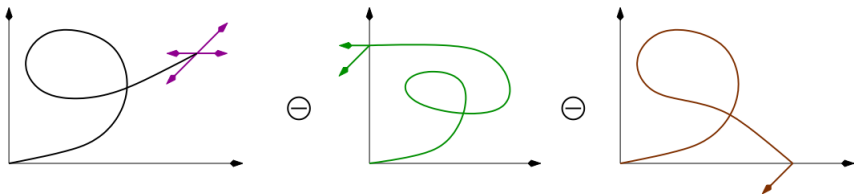
Task: Solve this functional equation!

Algebraic reformulation: solving a functional equation

Generating function: $G(t; x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^n \sum_{j=0}^n g(n; i, j) t^n x^i y^j \in \mathbb{Q}[x, y][[t]]$

“Kernel equation”:

$$G(t; x, y) = 1 + t \left(xy + x + \frac{1}{xy} + \frac{1}{x} \right) G(t; x, y) \\ - t \left(\frac{1}{x} + \frac{1}{xy} \right) G(t; 0, y) - t \frac{1}{xy} (G(t; x, 0) - G(t; 0, 0))$$



Task: For the other models: solve 78 similar equations!

Main results (I): algebraicity of Gessel walks

Theorem [Kreweras 1965; 100 pages long combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix} 1/3 & 2/3 & 1 \\ 3/2 & 2 \end{matrix} \middle| 27t^3\right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

Theorem [Kauers, Koutschan & Zeilberger 2009: former Gessel's conj. 1]

$$G(t;0,0) = {}_3F_2\left(\begin{matrix} 5/6 & 1/2 & 1 \\ 5/3 & 2 \end{matrix} \middle| 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

Question: What about the structure of $K(t; x, y)$ and $G(t; x, y)$?

Main results (I): algebraicity of Gessel walks

Theorem [Kreweras 1965; 100 pages long combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix} 1/3 & 2/3 & 1 \\ 3/2 & 2 \end{matrix} \middle| 27t^3\right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

Theorem [Kauers, Koutschan & Zeilberger 2009: former Gessel's conj. 1]

$$G(t;0,0) = {}_3F_2\left(\begin{matrix} 5/6 & 1/2 & 1 \\ 5/3 & 2 \end{matrix} \middle| 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

Question: What about the structure of $K(t; x, y)$ and $G(t; x, y)$?

Theorem [Gessel 1986, Bousquet-Mélou 2005] $K(t; x, y)$ is algebraic.

Theorem [B. & Kauers 2010: former Gessel's conj. 2] $G(t; x, y)$ is algebraic.

Main results (I): algebraicity of Gessel walks

Theorem [Kreweras 1965; 100 pages long combinatorial proof!]

$$K(t; 0, 0) = {}_3F_2 \left(\begin{matrix} 1/3 & 2/3 & 1 \\ 3/2 & 2 \end{matrix} \middle| 27t^3 \right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

Theorem [Kauers, Koutschan & Zeilberger 2009: former Gessel's conj. 1]

$$G(t; 0, 0) = {}_3F_2 \left(\begin{matrix} 5/6 & 1/2 & 1 \\ 5/3 & 2 \end{matrix} \middle| 16t^2 \right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

Question: What about the structure of $K(t; x, y)$ and $G(t; x, y)$?

Theorem [Gessel 1986, Bousquet-Mélou 2005] $K(t; x, y)$ is algebraic.

Theorem [B. & Kauers 2010: former Gessel's conj. 2] $G(t; x, y)$ is algebraic.

- ▷ Computer-driven discovery and proof.
- ▷ **Guess'n'Prove** method, using **Hermite-Padé approximants**[†] → Yesterday

[†] Minimal polynomial $P(x, y, t, G(t; x, y)) = 0$ has $> 10^{11}$ terms; ≈ 30 Gb (!)

Main results (I): algebraicity of Gessel walks

Theorem [Kreweras 1965; 100 pages long combinatorial proof!]

$$K(t; 0, 0) = {}_3F_2 \left(\begin{matrix} 1/3 & 2/3 & 1 \\ & 3/2 & 2 \end{matrix} \middle| 27t^3 \right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

Theorem [Kauers, Koutschan & Zeilberger 2009: former Gessel's conj. 1]

$$G(t; 0, 0) = {}_3F_2 \left(\begin{matrix} 5/6 & 1/2 & 1 \\ & 5/3 & 2 \end{matrix} \middle| 16t^2 \right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

Question: What about the structure of $K(t; x, y)$ and $G(t; x, y)$?

Theorem [Gessel 1986, Bousquet-Mélou 2005] $K(t; x, y)$ is algebraic.

Theorem [B. & Kauers 2010: former Gessel's conj. 2] $G(t; x, y)$ is algebraic.

- ▷ Computer-driven discovery and proof.
- ▷ **Guess'n'Prove** method, using **Hermite-Padé approximants**[†] → Yesterday
- ▷ New (human) proofs [B., Kurkova & Raschel 2013], [Bousquet-Mélou 2015]

[†] Minimal polynomial $P(x, y, t, G(t; x, y)) = 0$ has $> 10^{11}$ terms; ≈ 30 Gb (!)

Main results (II): Explicit form for $G(t; x, y)$

Theorem [B., Kauers & van Hoeij 2010]

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \dots$ be a root of

$$(V - 1)(1 + 3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2 + 83)t^6 + \dots$ be a root of

$$\begin{aligned} & x(V - 1)(V + 1)U^3 - 2V(3x + 5xV - 8Vt)U^2 \\ & - xV(V^2 - 24V - 9)U + 2V^2(xV - 9x - 8Vt) = 0, \end{aligned}$$

let $W = t^2 + (y + 8)t^4 + 2(y^2 + 8y + 41)t^6 + \dots$ be a root of

$$y(1 - V)W^3 + y(V + 3)W^2 - (V + 3)W + V - 1 = 0.$$

Then $G(t; x, y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t - xy} - \frac{1}{tx(y+1)}.$$

▷ Computer-driven discovery and proof; no human proof yet.

Main results (II): Explicit form for $G(t; x, y)$

Theorem [B., Kauers & van Hoeij 2010]

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \dots$ be a root of

$$(V - 1)(1 + 3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2 + 83)t^6 + \dots$ be a root of

$$\begin{aligned} & x(V - 1)(V + 1)U^3 - 2V(3x + 5xV - 8Vt)U^2 \\ & - xV(V^2 - 24V - 9)U + 2V^2(xV - 9x - 8Vt) = 0, \end{aligned}$$

let $W = t^2 + (y + 8)t^4 + 2(y^2 + 8y + 41)t^6 + \dots$ be a root of

$$y(1 - V)W^3 + y(V + 3)W^2 - (V + 3)W + V - 1 = 0.$$

Then $G(t; x, y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t - xy} - \frac{1}{tx(y+1)}.$$

- ▷ Computer-driven discovery and proof; no human proof yet.
- ▷ Proof uses **guessed minimal polynomials** for $G(t; x, 0)$ and $G(t; 0, y)$.

Main results (II): Explicit form for $G(t; x, y)$

Theorem [B., Kauers & van Hoeij 2010]

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \dots$ be a root of

$$(V - 1)(1 + 3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2 + 83)t^6 + \dots$ be a root of

$$\begin{aligned} & x(V - 1)(V + 1)U^3 - 2V(3x + 5xV - 8Vt)U^2 \\ & - xV(V^2 - 24V - 9)U + 2V^2(xV - 9x - 8Vt) = 0, \end{aligned}$$

let $W = t^2 + (y + 8)t^4 + 2(y^2 + 8y + 41)t^6 + \dots$ be a root of

$$y(1 - V)W^3 + y(V + 3)W^2 - (V + 3)W + V - 1 = 0.$$

Then $G(t; x, y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t - xy} - \frac{1}{tx(y+1)}.$$

- ▷ Computer-driven discovery and proof; ~~no human proof yet~~
- ▷ Recent (human) proofs [B., Kurkova, Raschel '13], [Bousquet-Mélou '15]

Main results (III): Conjectured D-Finite $F(t; 1, 1)$ [B. & Kauers 2009]

	OEIS	\mathfrak{G}	Pol size	ODE size		OEIS	\mathfrak{G}	Pol size	ODE size
1	A005566		—	3, 4	13	A151275		—	5, 24
2	A018224		—	3, 5	14	A151314		—	5, 24
3	A151312		—	3, 8	15	A151255		—	4, 16
4	A151331		—	3, 6	16	A151287		—	5, 19
5	A151266		—	5, 16	17	A001006		2, 2	2, 3
6	A151307		—	5, 20	18	A129400		2, 2	2, 3
7	A151291		—	5, 15	19	A005558		—	3, 5
8	A151326		—	5, 18					
9	A151302		—	5, 24	20	A151265		6, 8	4, 9
10	A151329		—	5, 24	21	A151278		6, 8	4, 12
11	A151261		—	4, 15	22	A151323		4, 4	2, 3
12	A151297		—	5, 18	23	A060900		8, 9	3, 5

Equation sizes = {order, degree}@{algeq, diffeq}

- ▷ Computerized discovery by enumeration + Hermite–Padé
- ▷ 1–22: Confirmed by human proofs in [Bousquet-Mélou & Mishna 2010]
- ▷ 23: Confirmed by a human proof in [B., Kurkova & Raschel 2015]

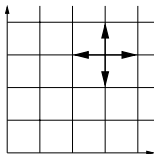
Main results (III): Conjectured D-Finite $F(t; 1, 1)$ [B. & Kauers 2009]

	OEIS	\mathfrak{G}	alg	asympt		OEIS	\mathfrak{G}	alg	asympt
1	A005566		N	$\frac{4}{\pi} \frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi} \frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi} \frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi} \frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi} \frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi} \frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi} \frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}} \frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi} \frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}} \frac{6^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
9	A151302		N	$\frac{1}{3} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3} \sqrt{\frac{7}{3\pi}} \frac{7^n}{n^{1/2}}$	22	A151323		Y	$\frac{\sqrt{23}^{3/4}}{\Gamma(1/4)} \frac{6^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)} \frac{4^n}{n^{2/3}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi} \frac{(2B)^n}{n^2}$					

$$A = 1 + \sqrt{2}, \quad B = 1 + \sqrt{3}, \quad C = 1 + \sqrt{6}, \quad \lambda = 7 + 3\sqrt{6}, \quad \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$$

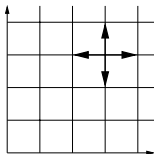
- ▶ Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.
- ▶ Confirmed by human proofs in [Melzer & Wilson, 2015]

The group of a model: the simple walk case



The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$

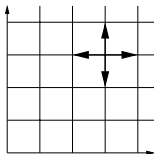
The group of a model: the simple walk case



The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x, y) = \left(x, \frac{1}{y}\right), \quad \phi(x, y) = \left(\frac{1}{x}, y\right),$$

The group of a model: the simple walk case



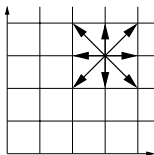
The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x, y) = \left(x, \frac{1}{y}\right), \quad \phi(x, y) = \left(\frac{1}{x}, y\right),$$

and thus under any element of the group

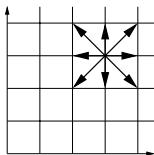
$$\langle \psi, \phi \rangle = \left\{ (x, y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

The group of a model: the general case



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j) \in \mathfrak{S}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$

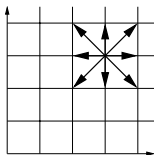
The group of a model: the general case



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j) \in \mathfrak{S}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$ is left invariant under

$$\psi(x, y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right),$$

The group of a model: the general case



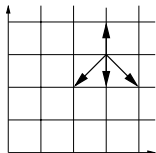
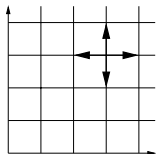
The polynomial $\chi_{\mathfrak{G}} := \sum_{(i,j) \in \mathfrak{G}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$ is left invariant under

$$\psi(x, y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right),$$

and thus under any element of the group

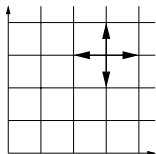
$$\mathcal{G}_{\mathfrak{G}} := \langle \psi, \phi \rangle.$$

Examples of groups

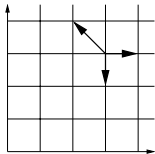


Order 4,

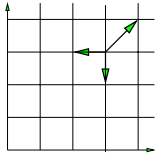
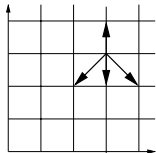
Examples of groups



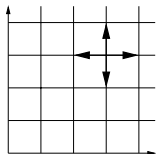
Order 4,



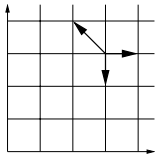
order 6,



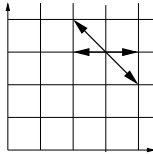
Examples of groups



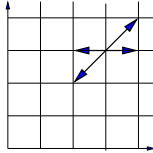
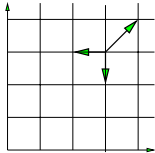
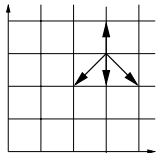
Order 4,



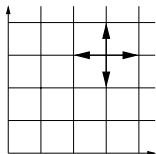
order 6,



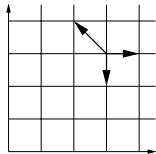
order 8,



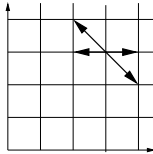
Examples of groups



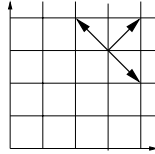
Order 4,



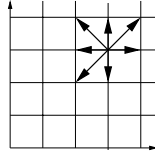
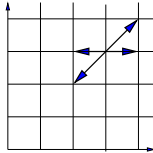
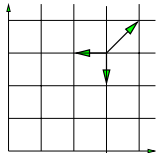
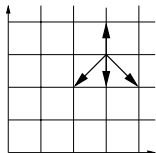
order 6,



order 8,



order ∞ .



An important concept: the orbit sum (OS)

The **orbit sum of a model** \mathfrak{G} is the following polynomial in $\mathbb{Q}[x, x^{-1}, y, y^{-1}]$:

$$\text{OrbitSum}(\mathfrak{G}) := \sum_{\theta \in \mathcal{G}_{\mathfrak{G}}} (-1)^{\theta} \theta(xy)$$

▷ E.g., for the simple walk:

$$\text{OS} \begin{array}{c} \nearrow \\ \leftarrow \\ \downarrow \\ \rightarrow \\ \nwarrow \end{array} = x \cdot y - \frac{1}{x} \cdot y + \frac{1}{x} \cdot \frac{1}{y} - x \cdot \frac{1}{y}$$

▷ For 4 models, the orbit sum is zero:

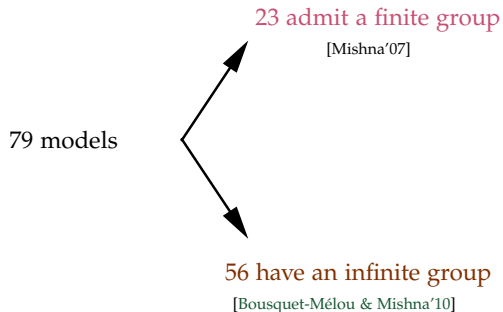


E.g. for the **Kreweras** model:

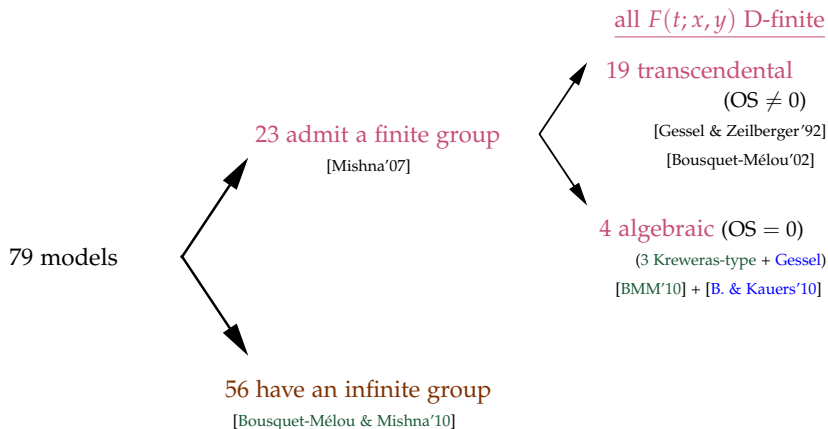
$$\text{OS} \begin{array}{c} \nearrow \\ \leftarrow \\ \downarrow \\ \rightarrow \\ \nwarrow \end{array} = x \cdot y - \frac{1}{xy} \cdot y + \frac{1}{xy} \cdot x - y \cdot x + y \cdot \frac{1}{xy} - x \cdot \frac{1}{xy} = 0$$

79 models

The 79 models: finite and infinite groups

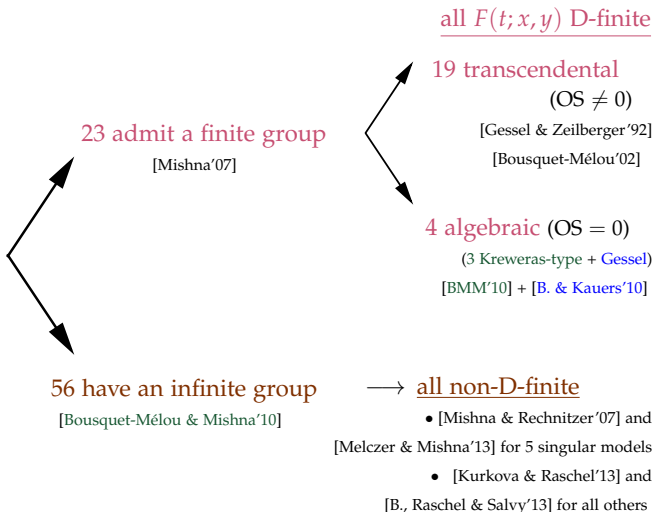


The 79 models: finite and infinite groups



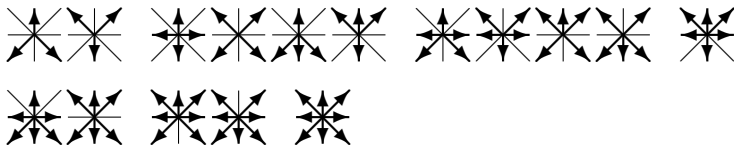
The 79 models: finite and infinite groups

79 models



The 23 models with a finite group

(i) 16 with a **vertical symmetry**, and group isomorphic to D_2



(ii) 5 with a **diagonal** or **anti-diagonal symmetry**, and group isomorphic to D_3



(iii) 2 with group isomorphic to D_4



(i): vertical symmetry; (ii)+(iii): zero drift $\sum_{s \in \mathfrak{G}} s$

In **red**, models with $OS = 0$ and **algebraic GF**

Main results (IV): explicit expressions for the 19 D-finite transcendental models

Theorem [B., Chyzak, van Hoeij, Kauers & Pech, 2016]

Let \mathfrak{S} be one of the 19 models with finite group $\mathcal{G}_{\mathfrak{S}}$, and non-zero orbit sum. Then

- $F_{\mathfrak{S}}$ is expressible using iterated integrals of ${}_2F_1$ expressions.
- Among the 19×4 specializations of $F_{\mathfrak{S}}(t; x, y)$ at $(x, y) \in \{0, 1\}^2$, only 4 are algebraic: for $\mathfrak{S} = \begin{matrix} \cdot & \cdot \\ \nearrow & \searrow \\ \cdot & \cdot \end{matrix}$ at $(1, 1)$, and $\mathfrak{S} = \begin{matrix} \cdot & \cdot \\ \nwarrow & \nearrow \\ \cdot & \cdot \end{matrix}$ at $(1, 0), (0, 1), (1, 1)$

Main results (IV): explicit expressions for the 19 D-finite transcendental models

Theorem [B., Chyzak, van Hoeij, Kauers & Pech, 2016]

Let \mathfrak{S} be one of the 19 models with finite group $\mathcal{G}_{\mathfrak{S}}$, and non-zero orbit sum. Then

- $F_{\mathfrak{S}}$ is expressible using iterated integrals of ${}_2F_1$ expressions.
- Among the 19×4 specializations of $F_{\mathfrak{S}}(t; x, y)$ at $(x, y) \in \{0, 1\}^2$, only 4 are algebraic: for $\mathfrak{S} = \begin{matrix} \nearrow \\ \uparrow \\ \nwarrow \end{matrix}$ at $(1, 1)$, and $\mathfrak{S} = \begin{matrix} \nwarrow \\ \uparrow \\ \searrow \end{matrix}$ at $(1, 0), (0, 1), (1, 1)$

Example (King walks in the quarter plane, A025595)

$$F_{\begin{matrix} \nwarrow \\ \uparrow \\ \searrow \end{matrix}}(t; 1, 1) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\frac{3}{2} \quad \frac{3}{2} \mid \frac{16x(1+x)}{(1+4x)^2}\right) dx$$
$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \dots$$

Main results (IV): explicit expressions for the 19 D-finite transcendental models

Theorem [B., Chyzak, van Hoeij, Kauers & Pech, 2016]

Let \mathfrak{S} be one of the 19 models with finite group $\mathcal{G}_{\mathfrak{S}}$, and non-zero orbit sum. Then

- $F_{\mathfrak{S}}$ is expressible using iterated integrals of ${}_2F_1$ expressions.
- Among the 19×4 specializations of $F_{\mathfrak{S}}(t; x, y)$ at $(x, y) \in \{0, 1\}^2$, only 4 are algebraic: for $\mathfrak{S} = \begin{matrix} \nearrow \\ \leftarrow \\ \searrow \\ \rightarrow \end{matrix}$ at $(1, 1)$, and $\mathfrak{S} = \begin{matrix} \nwarrow \\ \nearrow \\ \swarrow \\ \searrow \end{matrix}$ at $(1, 0), (0, 1), (1, 1)$

Example (King walks in the quarter plane, A025595)

$$F_{\begin{matrix} \nwarrow \\ \nearrow \\ \swarrow \\ \searrow \end{matrix}}(t; 1, 1) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\frac{3}{2}, \frac{3}{2} \mid \frac{16x(1+x)}{(1+4x)^2}\right) dx$$
$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \dots$$

- ▷ Computer-driven discovery and proof; no human proof yet.
- ▷ Proof uses **creative telescoping**, **ODE factorization**, **ODE solving**.

Hypergeometric Series Occurring in Explicit Expressions for $F(t; 1, 1)$

	hyp ₁	hyp ₂	w		hyp ₁	hyp ₂	w
1	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{3}{2} \\ 2 \end{matrix} \middle w\right)$	$16t^2$	10	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{9}{4} & \frac{11}{4} \\ 3 \end{matrix} \middle w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
2	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle w\right)$		$16t^2$	11	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{3}{2} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{5}{2} \\ 3 \end{matrix} \middle w\right)$	$\frac{16t^2}{4t^2+1}$
3	${}_2F_1\left(\begin{matrix} \frac{3}{2} & \frac{3}{2} \\ 2 \end{matrix} \middle w\right)$		$\frac{16t}{(2t+1)(6t+1)}$	12	${}_2F_1\left(\begin{matrix} \frac{5}{4} & \frac{7}{4} \\ 1 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{5}{4} & \frac{7}{4} \\ 2 \end{matrix} \middle w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
4	${}_2F_1\left(\begin{matrix} \frac{3}{2} & \frac{3}{2} \\ 2 \end{matrix} \middle w\right)$		$\frac{16t(1+t)}{(1+4t)^2}$	13	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 3 \end{matrix} \middle w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
5	${}_2F_1\left(\begin{matrix} \frac{3}{4} & \frac{5}{4} \\ 1 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{5}{4} & \frac{7}{4} \\ 2 \end{matrix} \middle w\right)$	$64t^4$	14	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{9}{4} & \frac{11}{4} \\ 3 \end{matrix} \middle w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
6	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 3 \end{matrix} \middle w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$	15	${}_2F_1\left(\begin{matrix} \frac{1}{4} & \frac{3}{4} \\ 1 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{3}{4} & \frac{5}{4} \\ 2 \end{matrix} \middle w\right)$	$64t^4$
7	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{3}{2} \\ 1 \end{matrix} \middle w\right)$	$\frac{16t^2}{4t^2+1}$	16	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{9}{4} & \frac{11}{4} \\ 3 \end{matrix} \middle w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$
8	${}_2F_1\left(\begin{matrix} \frac{5}{4} & \frac{7}{4} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$	19	${}_2F_1\left(-\frac{1}{2} \middle \frac{1}{2} \middle w\right)$	${}_2F_1\left(\frac{1}{2} \middle \frac{1}{2} \middle w\right)$	$16t^2$
9	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 3 \end{matrix} \middle w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$				

▷ All related to complete elliptic integrals!

Theorem [B., Rachel & Salvy 2013]

Let \mathfrak{G} be one of the 51 non-singular models with infinite group $\mathcal{G}_{\mathfrak{G}}$.
Then $F_{\mathfrak{G}}(t; 0, 0)$, and in particular $F_{\mathfrak{G}}(t; x, y)$, are non-D-finite.

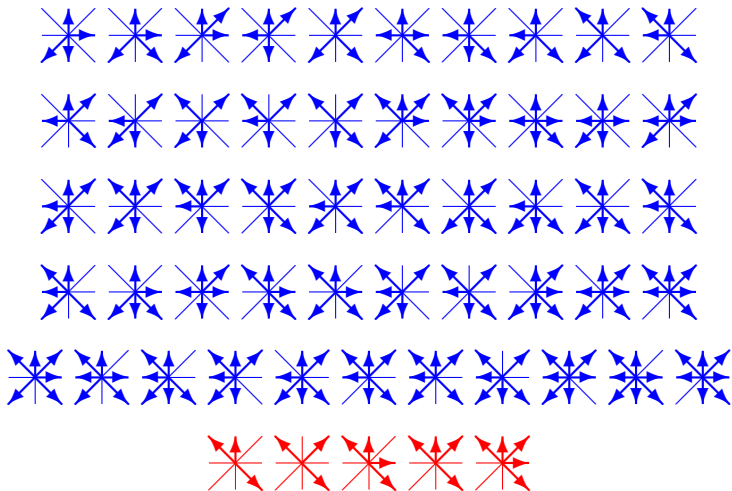
Theorem [B., Rachel & Salvy 2013]

Let \mathfrak{G} be one of the 51 non-singular models with infinite group $\mathcal{G}_{\mathfrak{G}}$.
Then $F_{\mathfrak{G}}(t; 0, 0)$, and in particular $F_{\mathfrak{G}}(t; x, y)$, are non-D-finite.

- ▷ **Algorithmic proof.** Uses **Gröbner basis computations, polynomial factorization, cyclotomy testing.**
- ▷ Based on **two ingredients: asymptotics + irrationality.**

- ▷ [Kurkova & Raschel 2013] Human proof that $F_{\mathfrak{G}}(t; x, y)$ is non-D-finite.
- ▷ No human proof yet for $F_{\mathfrak{G}}(t; 0, 0)$ **non-D-finite.**

The 56 models with infinite group



In **blue**, non-singular models, solved by [B., Raschel & Salvy 2013]

In **red**, singular models, solved by [Melczer & Mishna 2013]

Example: the scarecrows

[B., Raschel & Salvy 2013]: $F_{\mathfrak{G}}(t;0,0)$ is not D-finite for the models



For the 1st and the 3rd, the excursions sequence $[t^n] F_{\mathfrak{G}}(t;0,0)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \dots$$

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\dots$

The **irrationality** of α prevents $F_{\mathfrak{G}}(t;0,0)$ from being D-finite.

The Main Theorem Let \mathfrak{S} be one of the 74 non-singular models. The following assertions are equivalent:

- (1) The full generating series $F_{\mathfrak{S}}(t; x, y)$ is D-finite
- (2) the excursions generating series $F_{\mathfrak{S}}(t; 0, 0)$ is D-finite
- (3) the excursions sequence $[t^n] F_{\mathfrak{S}}(t; 0, 0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$
- (4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^* \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)
- (5) the step set \mathfrak{S} has either an axial symmetry, or zero drift and cardinal different from 5.

Summary: Classification of 2D non-singular walks

The Main Theorem Let \mathfrak{S} be one of the 74 non-singular models. The following assertions are equivalent:

- (1) The full generating series $F_{\mathfrak{S}}(t; x, y)$ is **D-finite**
- (2) the excursions generating series $F_{\mathfrak{S}}(t; 0, 0)$ is **D-finite**
- (3) the excursions sequence $[t^n] F_{\mathfrak{S}}(t; 0, 0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$
- (4) the group $\mathcal{G}_{\mathfrak{S}}$ is **finite** (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^* \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)
- (5) the step set \mathfrak{S} has either an **axial symmetry**, or **zero drift and cardinal different from 5**.

Moreover, under (1)–(5), $F_{\mathfrak{S}}(t; x, y)$ is **algebraic** if and only if the model \mathfrak{S} has **positive covariance** $\sum_{(i,j) \in \mathfrak{S}} ij - \sum_{(i,j) \in \mathfrak{S}} i \cdot \sum_{(i,j) \in \mathfrak{S}} j > 0$, and iff it has **OS = 0**.

Summary: Classification of 2D non-singular walks

The Main Theorem Let \mathfrak{S} be one of the 74 non-singular models. The following assertions are equivalent:

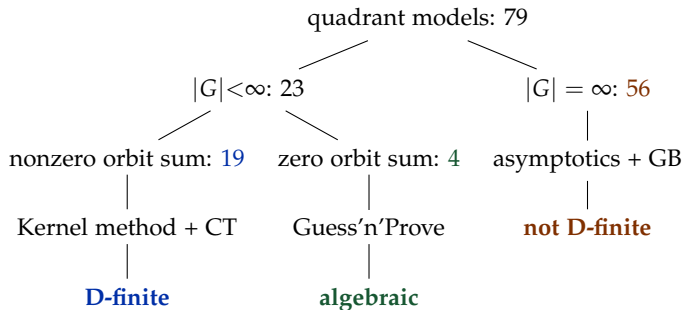
- (1) The full generating series $F_{\mathfrak{S}}(t; x, y)$ is **D-finite**
- (2) the excursions generating series $F_{\mathfrak{S}}(t; 0, 0)$ is **D-finite**
- (3) the excursions sequence $[t^n] F_{\mathfrak{S}}(t; 0, 0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$
- (4) the group $\mathcal{G}_{\mathfrak{S}}$ is **finite** (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^* \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)
- (5) the step set \mathfrak{S} has either an **axial symmetry**, or **zero drift and cardinal different from 5**.

Moreover, under (1)–(5), $F_{\mathfrak{S}}(t; x, y)$ is **algebraic** if and only if the model \mathfrak{S} has **positive covariance** $\sum_{(i,j) \in \mathfrak{S}} ij - \sum_{(i,j) \in \mathfrak{S}} i \cdot \sum_{(i,j) \in \mathfrak{S}} j > 0$, and iff it has **OS = 0**.

In this case, $F_{\mathfrak{S}}(t; x, y)$ is expressible using **nested radicals**.

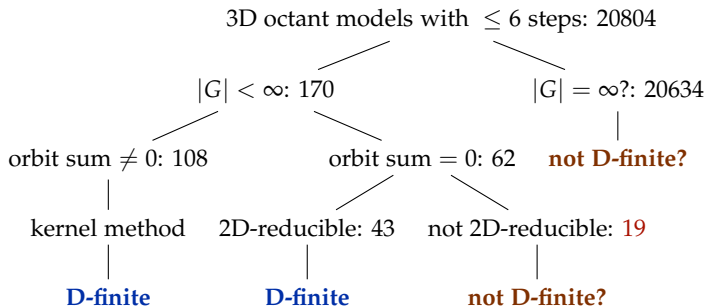
If not, $F_{\mathfrak{S}}(t; x, y)$ is expressible using **iterated integrals of ${}_2F_1$ expressions**.

Summary: Walks with unit steps in \mathbb{N}^2



Extensions: Walks with unit steps in \mathbb{N}^3

$2^{3^3-1} \approx 67$ millions models, of which ≈ 11 million inherently 3D

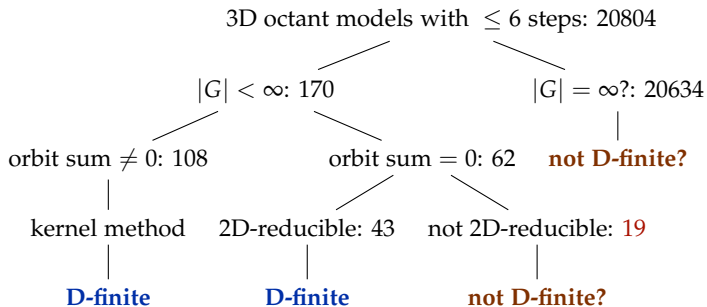


[B., Bousquet-Mélou, Kauers, Melczer 2015]

- ▷ Open question: **are there non-D-finite models with a finite group?**

Extensions: Walks with unit steps in \mathbb{N}^3

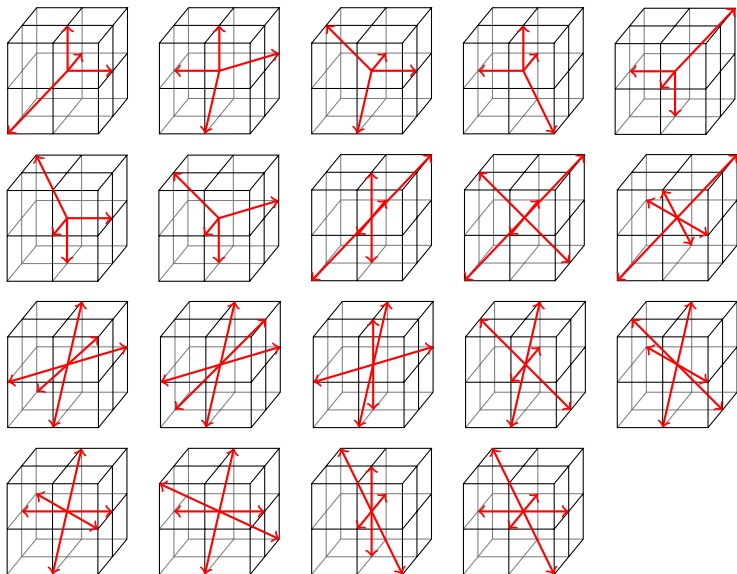
$2^{3^3-1} \approx 67$ millions models, of which ≈ 11 million inherently 3D

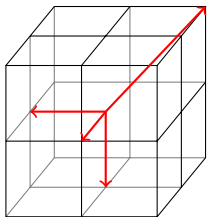


[B., Bousquet-Mélou, Kauers, Melczer 2015]

- ▷ Open question: **are there non-D-finite models with a finite group?**
- ▷ [Du, Hu, Wang, 2015]: proofs that groups are infinite in the 20634 cases
- ▷ [Bacher, Kauers, Yatchak, 2016]: extension to all 3D models; **170** models found with $|G| < \infty$ and orbit sum 0 (instead of **19**)

The 19 mysterious 3D-models





Two different computations suggest:

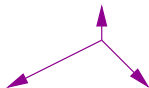
$$k_n \approx C \cdot 256^n / n^{3.3257570041744\dots},$$

so excursions are very probably transcendental
(and even non-D-finite)

Extensions: Walks in \mathbb{N}^2 with longer steps

- Define (and use) a group \mathcal{G} for models with larger steps?
- **Example:** When $\mathfrak{S} = \{(0,1), (1,-1), (-2,-1)\}$, there is an underlying group that is finite and

$$xyF(t; x, y) = [x^{>0}y^{>0}] \frac{(x - 2x^{-2})(y - (x - x^{-2})y^{-1})}{1 - t(xy^{-1} + y + x^{-2}y^{-1})}$$



[B., Bousquet-Mélou & Melczer, in preparation]

- ▷ Current status:
- 680 models with one large step, 643 **proved non D-finite**, 32 of 37 have differential equations **guessed**.
 - 5910 models with two large steps, 5754 **proved non D-finite**, 69 of 156 have differential equations **guessed**.

Conclusion



Computer algebra may solve difficult combinatorial problems



Classification of $F(t; x, y)$ **fully completed** for 2D small step walks



Robust algorithmic methods, based on efficient algorithms:

- **Guess'n'Prove**
- **Creative Telescoping**



Brute-force and/or use of naive algorithms = **hopeless**.

E.g. size of algebraic equations for $G(t; x, y) \approx 30\text{Gb}$.

Conclusion



Computer algebra may solve difficult combinatorial problems



Classification of $F(t; x, y)$ **fully completed** for 2D small step walks



Robust algorithmic methods, based on efficient algorithms:

- **Guess'n'Prove**
- **Creative Telescoping**



Brute-force and/or use of naive algorithms = **hopeless**.

E.g. size of algebraic equations for $G(t; x, y) \approx 30\text{Gb}$.



Lack of “purely human” proofs for some results.



Still missing a unified proof of: **finite group** \leftrightarrow **D-finite**.



Open: is $F(t; 1, 1)$ **non-D-finite** for all 56 models with infinite group?



Many open questions in dimension > 2 .

- [Automatic classification of restricted lattice walks](#), with M. Kauers. Proc. FPSAC, 2009.
- [The complete generating function for Gessel walks is algebraic](#), with M. Kauers. Proc. Amer. Math. Soc., 2010.
- [Explicit formula for the generating series of diagonal 3D Rook paths](#), with F. Chyzak, M. van Hoeij and L. Pech. Séminaire Lotharingien de Combinatoire, 2011.
- [Non-D-finite excursions in the quarter plane](#), with K. Raschel and B. Salvy. Journal of Combinatorial Theory A, 2013.
- [On 3-dimensional lattice walks confined to the positive octant](#), with M. Bousquet-Mélou, M. Kauers and S. Melczer. Annals of Comb., 2015.
- [A human proof of Gessel's lattice path conjecture](#), with I. Kurkova, K. Raschel, Transactions of the AMS, 2015.
- [Explicit Differentiably Finite Generating Functions of Walks with Small Steps in the Quarter Plane](#), with F. Chyzak, M. van Hoeij, M. Kauers and L. Pech, 2016.

Thanks for your attention!